Exercice - M0171C

1) et 2) Placons les points A d'affixe $z_A = 2$ et B d'affixe $z_B = 2e^{i\frac{3\pi}{4}}$, puis le point I milieu du segment $[AB]$.

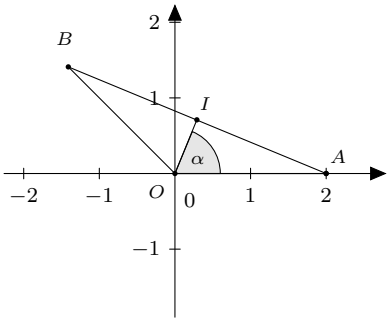

FIGURE 1 – Ligne trigonometrique $\frac{\pi}{8}$

3 Calculons l'affixe du point I. Nous avons

$$z_A = 2 \qquad z_B = 2\cos\left(\frac{3\pi}{4}\right) + i \cdot 2\sin\left(\frac{3\pi}{4}\right) = -\sqrt{2} + i\sqrt{2}$$

Nous en déduisons

$$z_I = \frac{z_A + z_B}{2} = \frac{2 - \sqrt{2} + i\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

Calculons la forme trigonométrique de z_i

$$|z_I| = \sqrt{\left(\frac{2-\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{4 - 4\sqrt{2} + 2 + 2}{4}} = \sqrt{2-\sqrt{2}}$$

Il vient

$$\begin{aligned} z_I &= \sqrt{2-\sqrt{2}}\left(\frac{2-\sqrt{2}}{2\sqrt{2-\sqrt{2}}} + i\frac{\sqrt{2}}{2\sqrt{2-\sqrt{2}}}\right) \\ &= \sqrt{2-\sqrt{2}}\left(\frac{\sqrt{2-\sqrt{2}}}{2} + i\frac{\sqrt{2}\sqrt{2+\sqrt{2}}}{2\sqrt{2-\sqrt{2}}\sqrt{2+\sqrt{2}}}\right) \\ &= \sqrt{2-\sqrt{2}}\left(\frac{\sqrt{2-\sqrt{2}}}{2} + i\frac{\sqrt{2}\sqrt{2+\sqrt{2}}}{2\sqrt{4-2}}\right) \\ &= \sqrt{2-\sqrt{2}}\left(\frac{\sqrt{2-\sqrt{2}}}{2} + i\frac{\sqrt{2+\sqrt{2}}}{2}\right) \end{aligned}$$

Or géométriquement, (OAB) forme un triangle isocèle et l'angle $(\vec{i}, \overrightarrow{OI})$ a pour mesure $\dfrac{3\pi}{8}$. Donc

$$z_i = \sqrt{2-\sqrt{2}}\left(\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)\right)$$

Donc

$$\cos\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2} \qquad \sin\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

Or

$$\cos\left(\frac{3\pi}{8}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \sin\left(\frac{\pi}{8}\right) \quad \text{et} \quad \sin\left(\frac{3\pi}{8}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \cos\left(\frac{\pi}{8}\right)$$

Conclusion

$$\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2} \qquad \text{et} \qquad \sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}$$